Rutgers University: Complex Variables and Advanced Calculus Written Qualifying Exam

January 2016: Problem 4 Solution

Exercise. Suppose f is an entire function with $\int\int_{\mathbb{C}} |f(z)|^2 dxdy < \infty$, Show taht $f(z) = 0$ for all $z \in \mathbb{C}$

Solution.

$$\int\int_{\mathbb{C}} |f(z)|^2 dxdy < \infty \quad \Longrightarrow \quad \int\int_{\mathbb{C}} |f(z)|^2 dxdy \text{ exists}$$
$$\Longrightarrow \quad |f(z)|^2 \text{ is bounded}$$
$$\Longrightarrow \quad |f(z)| \text{ is bounded}$$
$$\Longrightarrow \quad f(z) \text{ is bounded}$$

So f is a bounded, entire function $\Longrightarrow$ f is constant.

If $f(z) \neq 0$ then $|f(z)|^2 = c \in \mathbb{R}^+$

$$\Longrightarrow \int\int_{\mathbb{C}} |f(z)|^2 dxdy = c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 1 dxdy = \infty$$

So, $f(z) = 0$ for all $z \in \mathbb{C}$.